CONNECTEDNESS OF THE SET OF CENTRAL LYAPUNOV EXPONENTS

ABBAS FAKHARI[♮♯]

[♮]*School of Mathematics and Computer Science, Damghan University, 36715.364, Damghan, Iran,*
[♯]*School of Mathematics, Institute for Research In Fundamental Sciences (IPM), 19395-5746, Tehran, Iran.*
fakhari@du.ac.ir

Abstract. In this brief note we prove that there is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that for any $f \in \mathcal{R}$ and any partially hyperbolic homoclinic class $H(p, f)$ with one dimensional center direction, the set of central Lyapunov exponents associated to the ergodic with full support is an interval. Also, some remarks on the general case presented.

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1. Introduction

By definition, hyperbolic dynamical systems have nonzero Lyapunov exponents. However, they are not generic in the space of all dynamical systems. This necessitated weakening the notion of hyperbolicity. A wider class is that of partially hyperbolic systems studied by many authors (for a complete review, see [1-3]). A partially hyperbolic set may admit zero Lyapunov exponents along their central directions.

Question. Is the set of central Lyapunov vectors associated to ergodic measures on a partially hyperbolic set is convex?

In general, the answer is negative. In [4], the authors have studied destroying horseshoes via heterodimensional cycles. These maps are partially hyperbolic with a one dimensional center direction. They proved that every ergodic measure is hyperbolic, but the set of Lyapunov exponents in the central direction has a gap. However, there are examples of open sets of partially hyperbolic maps whose central Lyapunov exponents form a convex set (see for instance [5]). As known, the existence of heterodimensional cycles isn't a generic phenomenon. Now, a question may be arisen:

what can be said in generic mode?

There are partial answers to the question in the case of generic "locally maximal" or "Lyapunov stable" homoclinic classes [7, 8]. In this note, we give a positive answer to the question in the case of one dimensional centeral direction. In addition, we will prove that the support of the ergodic measure fills a gap equals to the whole of the homoclinic class.

Let M be a compact boundless Riemannian manifold and f be a diffeomorphism on it. For a periodic point p of f, we denote by $\pi(p)$ the period of p. For a hyperbolic periodic point p of f of period $\pi(p)$ the sets

$$W^s(p) = \{x \in M : f^{\pi(p)n}(x) \to p \ \ as\ n \to \infty\}, \text{ and}$$
$$W^u(p) = \{x \in M : f^{-\pi(p)n}(x) \to p \ \ as\ n \to \infty\};$$

are injectively immersed submanifolds of M. A point $x \in W^s(p) \cap W^u(p)$ is called a *homoclinic point* of f associated to p, and it is called a *transversal homoclinic point* of f if the above intersection is transversal at x. The closure of the transversal homoclinic points of f associated to the orbit of p is called the *homoclinic class* of f associated to p, and denoted by $H(p, f)$. Two saddles p and q are called *homoclinically related*, and write $p \sim q$, if

$$W^s(\mathcal{O}(p)) \overline{\pitchfork} W^u(\mathcal{O}(q)) \neq \phi \quad \text{and} \quad W^u(\mathcal{O}(p)) \overline{\pitchfork} W^s(\mathcal{O}(q)) \neq \phi.$$

Note that, by *Smale's transverse homoclinic point theorem* , $H(p, f)$ coincides with the closure of the set of all hyperbolic periodic points q with $p \sim q$.

Definition 1. A compact invariant set Λ is *partially hyperbolic* if it admits a splitting $T_\Lambda M = E^{ss} \oplus E^c \oplus E^{uu}$ such that

$$\|Df|_{E^{ss}}\| < \|Df|_{E^c}\| < \|Df|_{E^{uu}}\|,$$

furthermore, E^{ss} is uniformly contraction and E^{uu} is uniformly expansion; it means

$$\|Df|_{E^{ss}}\| < 1 < \|Df|_{E^{uu}}\|.$$

The bundles E^{ss} and E^{uu} of a partially hyperbolic set are always uniquely integrable. Denote by $W^{ss}_{loc}(x)$ (resp. $W^{uu}_{loc}(x)$) the *local strong stable* (resp. *unstable*) manifold of x tangent to $E^{ss}(x)$ (resp. $E^{uu}(x)$) at x ([1-3]). Recall that if a diffeomorphism f has a partially hyperbolic set Λ then there is an open neighborhood U of Λ and a neighborhood $\mathcal{U}$ of f such that the maximal invariant set $\tilde{\Lambda}_g$ of a diffeomorphism $g \in \mathcal{U}$ which is entirely contained in U is partially hyperbolic.

Denote by $\mathcal{M}_f(M)$ the set of f-invariant probability measures endowed with its usual topology; i.e., the unique metrizable topology such that $\mu_n \to \mu$ if and only if $\int f\, d\mu_n \to \int f\, d\mu$ for every continuous function $f : M \to \mathbb{R}$. The set of all ergodic elements of $\mathcal{M}_f(M)$ supported on an invariant set Λ is denoted by $\mathcal{M}_e(f|_\Lambda)$.

A point x in a partially hyperbolic set Λ is called *regular* if the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^n(x)|_{E^c(x)}\|$$

exists. By Oseledec's theorem, given any invariant measure μ the limits exist μ-a.e.. The limits above are called *central Lyapunov exponent* of x. When μ is ergodic the limit doesn't depend to initial point and is constant μ-a.e.. We recall that an invariant set Λ of a diffeomorphism f is *Lyapunov stable* if for any neighborhood U of Λ there is a V of it such that $f^n(V) \subset U$, for any $n \geq 0$. Surly, for any hyperbolic periodic point p in a Lyapunov stable invariant set Λ we have $W^u(p) \subset \Lambda$. The set Λ is *bi-Lyapunov stable* if it is Lyapunov stable for f and f^{-1}.

Theorem A. For a C^1-generic diffeomorphisms f and any partially hyperbolic homoclinic class $H(p, f)$ of f with the splitting $E^s \oplus E^c \oplus E^u$ such that

- either $\dim(E^c) = 1$ or
- $H(p, f)$ is bi-Lyapunov stable and E^c has a dominated splitting $E^c_1 \oplus E^c_2$ into one dimensional bundles.

the central Lyapunov exponents associated to the ergodic measures with support equal to $H(p, f)$ forms a convex set.

Problem 1. In general, is the set of Lyapunov vectors associated to the set of ergodic measures on a generic homoclinic class convex?

2. Proof of Theorem A

First, we state two following lemmas to guarantee the existence of hyperbolic periodic points with desired central Lyapunov exponents inside the homoclinic class.

Lemma 2. *(Pliss's Lemma [10]) Given $\lambda_0 < \lambda_1$ and $\mathcal{O}(p) \subset \tilde{\Lambda}$ there is some natural number m such that if $\prod_{j=0}^{t} \|Df|_{E^c(f^j(p_n))}\| \leq \lambda_0^t$ for some $t \geq m$ then there is a sequence $n_1 < n_2 < \cdots < n_k \leq t$ such that $\prod_{j=n_r}^{t} \|Df|_{E^c(f^j(p_n))}\| \leq \lambda_1^{t-n_r}$, for any $1 \leq r \leq k$.*

Lemma 3. *([11]) For any $0 < \lambda < 1$, there is $\epsilon > 0$ such that for $x \in \tilde{\Lambda}$ which satisfies $\prod_{j=0}^{n-1} \|Df|_{E^c(f^j(p_n))}\| \leq \lambda^n$, for all $n > 0$, we have diam$(f^n(W^{cs}_\epsilon(x)) \to 0$. In other word, the center stable manifold of x with size ϵ is in fact stable manifold.*

To prove Theorem A, we need to recall some C^1 generic statements. For a partially hyperbolic homoclinic class $H(p, f)$ and $\mu \in \mathcal{M}_e(f|_{H(p,f)})$, Let λ^c_μ be the Lyapunov exponent of μ along the central direction. For a hyperbolic periodic point $q \in H(p, f)$ of period $\pi(q)$, $\lambda^c_q = \log|\nu|/\pi(q)$ where $\nu = Df^{\pi(q)}(q)|_{E^c(q)}$. Put

$$\mathcal{LE}^c(H(p, f)) = \{\lambda^c_q;\, q \in H(p, f) \cap Per(f)\}$$

Proposition 4. *There is a residual subset $\mathcal{R}_0$ of Diff$^1(M)$ such that for any $f \in \mathcal{R}$ and any hyperbolic periodic point p of f,*

(1) *f is Kupka-Smale,*
(2) *the set $\mathcal{LE}^c(H(p,f))$ is connected ([7]),*
(3) *for any periodic point $q \in H(p,f)$ with index$(q) =$ index(p), we have $q \sim p$, ([8, 12]),*
(4) *for any ergodic measure μ of f, there is a sequence of periodic point p_n such that $\mu_{p_n} \to \mu$ in weak* topology and $\mathcal{O}(p_n) \to Supp(\mu)$ in Housdorff metric ([13]).*
(5) *If a homoclinic class $H(p,f)$ is bi-Lyapunov stable then for any sufficiently close g, $H(p_g, g)$ is bi-Lyapunov stable, where p_g is the continuation of p.*

Using arguments above we prove that any hyperbolic ergodic measure supported on a C^1-generic partially hyperbolic homoclinic class $H(p,f)$ can be approximated by periodic measures also supported on the homoclinic class. In fact, this is a special case of Bonatti's conjecture.

Bonatti's Conjecture. For a C^1-generic diffeomorphism f, any ergodic measure supported on a homoclinic class $H(p,f)$ of f can be approximate in weak topology by periodic measures supported on the homoclinic class.

Lemma 2.4, below, gives a positive answer to the conjecture in the case of hyperbolic measures on a partially hyperbolic homoclinic class.

Lemma 5. *For any $f \in \mathcal{R}_0$, any partially hyperbolic homoclinic class $H(p,f)$, any hyperbolic measure $\mu \in \mathcal{M}_e(f|_{H(p,f)})$ and any $\epsilon > 0$, there is a periodic point $q \in H(p,f)$ such that for any one dimensional center bundle E^c,*

$$|\lambda^c_p - \lambda^c_\mu| < \epsilon.$$

Proof. Choose $\lambda_0 = \lambda^c_\mu + \epsilon < \lambda_1 < 0$. By Proposition 2.3, for any $\mu \in \mathcal{M}_e(f|_{H(p,f)})$, there is a is a sequence $\{p_n\} \subset \tilde{\Lambda}$ such that $\lambda^c_{p_n} \to \lambda^c_\mu$. Hence, for sufficiently large n, $\lambda^c_{p_n} < \lambda_0 = \lambda^c_\mu + \epsilon < 0$ and so $\prod_{j=0}^{\pi(p_n)} |Df|_{E^c(f^j(p_n))}| \leq (\exp^{\lambda_0})^{\pi(p_n)}$. Now, by Pliss Lemma, for large n, there is a point $q_n \in \mathcal{O}(p_n)$ such that $\prod_{j=0}^{t} |Df|_{E^c(f^j(q_n))}| \leq (\exp^{\lambda_1})^t$ for any $t \leq \pi(p_n)$. By Lemma 2.2, the center-stable manifolds of q_n's are in fact stable manifolds and have uniform size. This implies that for large $n, m \in \mathbb{N}$, $p_n \sim p_m$ and $p_n \in H(p,f)$. $\square$

In what follows, we first state a lemma which is a modification of some recent results on C^1-generic dynamics in our context ([8]). The second part of the Lemma helps us to give a positive answer again to Bonatti's Conjecture in the case of ergodic measures, not necessarily hyperbolic, supported on bi-Lyapunov stable homoclinic class, Theorem 2.6 below. We recall that a periodic point p of a diffeomorphism f has *weak Lyapunov exponent* along a center bundle E^c if for some small $\delta > 0$, $|\lambda^c| < \delta$..

Lemma 6. *There is a residual subset $\mathcal{R}_1$ of Diff$^1(M)$ such that for any $f \in \mathcal{R}_1$ and any homoclinic class $H(p,f)$ of f having a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$ with non-hyperbolic center bundle E^c, we have*

- *if E^c is one-dimensional then there is a periodic point in the homoclinic class with weak Lyapunov exponent along E^c,*
- *if E^c has a dominated splitting $E^c = E_1^c \oplus E_2^c$ into one dimensional subbundles then either $H(p,f)$ contains a periodic point whose index is equal to $dim(E^s)$ or it contains a periodic point with weak Lyapunov exponent along E_1^c.*

Theorem 7. *For C^1-generic bi-Lyapunov stable homoclinic class $H(p,f)$, satisfying Proposition 2.3 and Lemma 2.5, any ergodic measure can be approximate by periodic measure inside the homoclinic class.*

Proof. As before, if an ergodic measure is hyperbolic then nothing remains. Hence, suppose that μ is a non-hyperbolic ergodic measure supported on the homoclinic class $H(p,f)$. Using the forth part of Proposition 2.3, one can find a sequence $\{p_n\}$ of periodic point such that $\mathcal{O}(p_n) \to Supp(\mu)$ in Housdorff metric and also the periodic measures associated to p_n tends toward μ in weak topology. Now, If the homoclinic class contains a periodic points q of index $s = \dim(E^s)$ then by the partial hyperbolicity, for large n, and positive iteretes m_n, $W^u(q) \cap W^s(f^{m_n}(p_n)) \neq \emptyset$. Hence, by the Lyapunov stability, $p_n \in H(p,f)$. Thus, suppose that for any periodic point in the homoclinic class its index equals to $s+1$.

By Lemma 2.5, there is periodic point q in the homoclinic class with weak Lyapunov exponents along the central direction E_1^c. We note that by the assumption $\text{index}(q) = \text{index}(p) = s+1$ and so $q \sim p$. Now, using Gourmelon's perturbation lemma [9], which allows us to perturb the derivative controlling the invariant manifolds, we can produce a periodic point q' of index s such that $W^s(p) \cap W^u(q') \neq \emptyset$. Again, by the Lyapunov stability, $q' \in H(p,f)$. This is contradiction. □

Now, we follow the approach suggested in [6] to provide sufficient conditions for the existence of desired ergodic invariant measure as a limit of periodic ones.

Definition 8. a periodic orbit X is a (γ, χ)-good approximation of a periodic orbit Y if the following holds

- for a subset Γ of X and a projection $\rho : \Gamma \to Y$,
$$d(f^j(x), f^j(\rho(x))) < \gamma$$
for any $x \in \Gamma$ and any $j = 0, 1, \ldots, \pi(Y) - 1$;
- $card(\Gamma)/card(X) \geq \chi$;
- $card(\rho^{-1}(y)$ is the same for all $y \in Y$.

The next lemma is a key point in the proof of the ergodicity of a limit measure. This lemma was suggested by Yu. Ilyashenko [6]. The following modified version, borrowed from [14], determines the support of the obtained ergodic measure.

Lemma 9. *Let $\{X_n\}$ be a sequence of periodic orbits with increasing periods $\pi(X_n)$ of f. Assume that there are two sequences of numbers $\{\gamma_n\}$, $\gamma_n > 0$ and $\{\chi_n\}$, $\chi_n \in (0,1]$, such that*

1. *for any $n \in \mathbb{N}$ the orbit X_{n+1} is a (γ_n, χ_n)-good approximation of X_n;*
2. $\sum_{n=1}^{\infty} \gamma_n < \infty$;
3. $\prod_{n=1}^{\infty} \chi_n \in (0, 1]$.

Then the sequence $\{\mu_{X_n}\}$ of atomic measures has a limit μ which is ergodic and

$$Supp(\mu) = \bigcap_{k=1}^{\infty} \overline{(\bigcup_{l=k}^{\infty} X_l)} \tag{1}$$

Lemma 10. *(lemma 3.5 in [14]) There is a residual subset $\mathcal{R}_2$ of $Diff^1(M)$ such that for any $f \in \mathcal{R}_2$, any partially hyperbolic homoclinic class $H(p, f)$, any one dimensional center bundle E^c, any saddle q in $H(p, f)$ and any $\epsilon > 0$, $H(p, f)$ contains a saddle q_1 which is homoclinically related to q, its orbit is ϵ-dense in the homoclinic class $H(p, f)$ and*

$$|\lambda_q^c - \lambda_{q_1}^c| < \epsilon.$$

Proof of Theorem A. We first proceed with the first case of the theorem. Let $f \in \mathcal{R}_0 \cap \mathcal{R}_1 \cap \mathcal{R}_2$ and $H(p, f)$ be a partially hyperbolic homoclinic class of f with one dimensional center direction. Put

$$\lambda_{min}^c = \min \lambda^c(\mu), \ \lambda_{max}^c = \max \lambda^c(\mu),$$

where "min" and "max" are given over all f-invariant ergodic measures supported on $H(P, f)$. We consider three possible cases.

(i) $\lambda_{min}^c < 0 < \lambda_{max}^c$. Let $s \in (\lambda_{min}^c, \lambda_{max}^c)$. We use Lemma 2.6 to construct an ergodic measure μ with $Supp(\mu) = H(p, f)$ such that $\mu_\mu^c = s$. By Lemma 2.7, there are two saddles p and q_0 such that $\lambda_p^c < 0 < s < \lambda_{q_0}^c$. Two cases may be occurred

• $s = 0$. This case is deduced from [14]. In fact, the authors in [14] have proved that if a C^1-generic homoclinic class which has partial splitting with one dimensional center direction has periodic points with different index then it can admit a non-hyperbolic ergodic measure whose support equals to the whole of the homoclinic class.

• $s \neq 0$. Suppose that $s > 0$. By the second item of Proposition 2.3, there is a periodic point $p_0 \in H(p, f)$ such that $0 < \lambda_{p_0}^c < s < \lambda_{q_0}^c$. Inductively, we find two sequences of $\{p_n\}$ and $\{q_n\}$ of the same index such that

1. $0 < \lambda_{p_{n-1}}^c < \lambda_{p_n}^c < \big(s + \lambda_{p_{n-1}}^c\big)/2 < s < \lambda_{q_n}^c < (s + \lambda_{q_{n-1}}^c)/2 < \lambda_{q_{n-1}}^c$,
2. q_n is $\epsilon/2^n$-dense in $H(p, f)$,
3. q_n is $(\epsilon/2^n, (1 - c/2^n))$-good approximation of q_{n-1}, for some positive constant c.

Let $0 < \lambda_{p_{n-1}}^c < s < \lambda_{q_{n-1}}^c$. By Lemma 2.7, there is a saddle $p_n \in H(p, f)$ with $\epsilon/2^n$-dense orbit such that $0 < \lambda_{p_{n-1}}^c < \lambda_{p_n}^c < \big(s + \lambda_{p_{n-1}}^c\big)/2 < s$. Since p_n and q_{n-1} have the same index they are homoclinically related. Let

$$x \in W^s(\mathcal{O}(q_{n-1})) \,\overline{\pitchfork}\, W^u(\mathcal{O}(p_n)) \ \text{ and } \ y \in W^u(\mathcal{O}(q_{n-1})) \,\overline{\pitchfork}\, W^s(\mathcal{O}(p_n)).$$

Consider a locally maximal hyperbolic set Λ_n containing $\mathcal{O}(p_n) \cup \mathcal{O}(x) \cup \mathcal{O}(y) \cup \mathcal{O}(q_{n-1})$. For some suitable $m_n \in \mathbb{N}$, put

$$\mathcal{PO}_n = \{ \underbrace{\mathcal{O}(p_n)}_{t_n-\text{times}}, f^{-m_m}(x), \dots, f^{m_n}(x), \underbrace{\mathcal{O}(q_{n-1})}_{s_n-\text{times}}, f^{-m_n}(y), \dots, f^{m_n}(y)\},$$

where $2m_nM/\big(t_n\pi(p_n)\lambda^c_{p_n} + s_n\pi(q_{n-1})\lambda^c_{q_{n-1}} + 2mM\big) < \epsilon/2^n$. If $\mathcal{PO}_n$ is $\epsilon/2^n$-shadowed by a saddle q_n then $\pi(q_n) = t_n\pi(p_n) + s_n\pi(q_{n-1}) + 2m_n$ and for suitable t_n and s_n one has

$$s < \lambda^c_{q_n} \approx \frac{t_n\pi(p_n)\lambda^c_{p_n} + s_n\pi(q_{n-1})\lambda^c_{q_{n-1}} + 2m_nM}{t_n\pi(p_n) + s_n\pi(q_{n-1}) + 2m_n} < \frac{s + \lambda^c_{q_{n-1}}}{2},$$

where $M = \max_{x\in M} \log \|Df(x)\|$. It is't difficult to see that

$$\chi_n = \frac{s_n\pi(p_n)}{t_n\pi(q_{n-1}) + s_n\pi(p_n) + 2m} = 1 - \frac{t_n\pi(q_{n-1}) + 2m}{t_n\pi(q_{n-1}) + s_m\pi(p_n) + 2m} > 1 - c/2^n,$$

for some positive constant c. Now, the sequence $\{q_n\}$ satisfies in the assumption of Lemma 2.6. Hence, the weak* limit μ of the atomic measures μ_{q_n} is ergodic and

$$\lambda^c_\mu = \int \|Df|_{E^c}\|\, d\mu = \lim_{n\to\infty} \int \|Df|_{E^c}\|\, d\mu_{q_n} = \lim_{n\to\infty} \lambda^c_{q_n} = s.$$

Furthermore, by (1), $\mathrm{Supp}(\mu) = H(p, f)$.

(ii) $\lambda^c_{min} > 0$ or $\lambda^c_{min} < 0$. In this case, the central direction is hyperbolic ([15]).

(iii) $\lambda^c_{min} = 0$ or $\lambda^c_{max} = 0$. In this case by the first part of Lemma 2.5, the problem is reduced to the previous ones.

In the second case of the theorem, one should notice that by Theorem 2.6, any ergodic measure can be approximated by periodic measures supported on the homoclinic class. By the second part of Proposition 2.3, we can suppose that these sequence of periodic orbits are coindex one. Hence, the lemma turns to the first case. □

3. Remarks in the case $E^c = E^c_1 \oplus E^c_2$

In general, we don't know much about the topological properties of the set of ergodic measures supported on a homoclinic class. Since the central directions are one dimensional the Lyapunov exponents along them define a continues map with in weak-topology. As before, one can deduce that the closure of the set of central Lyapunov exponents associated to the hyperbolic ergodic measures forms a convex set. One should notice that by the domination, the Lyapunov exponent along E^c_1 is strictly less than the Lyapunov exponents along E^c_2. Hence, in the case of non-hyperbolic ergodic measures two disjoint cases may be occurred, either $\lambda^\mu_1 < 0 = \lambda^\mu_2$ or $\lambda^\mu_1 = 0 < \lambda^\mu_2$. The overall picture of the set of central Lyapunov exponents is illustrated in Figure 1.

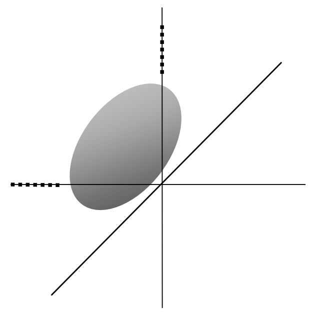

Figure 1. General situation of central Lyapunov exponents

The shadowed part is the closure of the set of central Lyapunov exponents associated to the hyperbolic ergodic measures supported on $H(p, f)$ which is in fact the set $\mathcal{LE}^c(H(p, f))$. Using the central model for chain transitive sets, S. Crovisier *et al.* have obtained the following version of the Mane's ergodic closing lemma inside the homoclinic class ([8]).

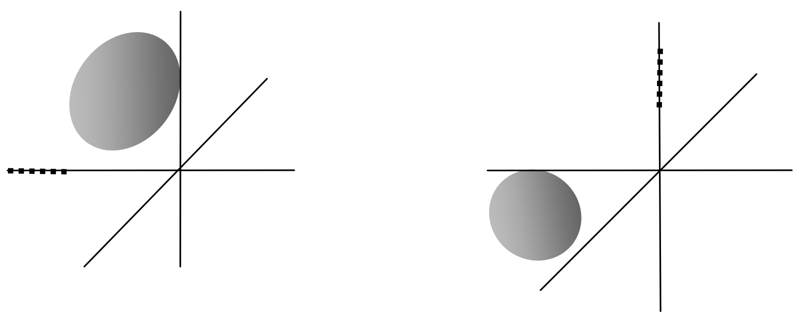

Figure 2. Special situations of central Lyapunov exponents

Proposition 11. *For any C^1-generic diffeomorphism f, let $H(p, f)$ be a partially hyperbolic homoclinic class with one dimensional central bundles. Let i be the minimal stable dimension of its periodic orbits. If $H(p, f)$ contains periodic point with i^{th} weak Lyapunov exponent, then for any ergodic measure supported on $H(p, f)$ whose $(i-1)^{th}$ Lyapunov exponent is zero, there exists periodic orbits contained in $H(p, f)$ whose associated measures converge for the weak topology towards the measure.*

The lemma suggests two simplest situations showed in the Figure 2.

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